USING CHAOS TO OBTAIN GLOBAL SOLUTIONS IN COMPUTATIONAL KINEMATICS

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ABSTRACT

In this paper we examine the sensitive dependence on the initial conditions of the Newton-Raphson search technique. It is demonstrated that this sensitivity has a fractal nature which can be effectively utilized to find all solutions to a nonlinear equation. The developed technique uses an important feature of fractals to preserve shape of basins of attraction on infinitely small scales.

1. INTRODUCTION

"... it may happen that slight differences in the initial conditions produce very great differences in the final phenomena: a slight error in the former would make an enormous error in the latter. Prediction becomes impossible..." H. Poincaré

Poincaré [1921] and Lorenz [1962] showed that behavior of a deterministic system needs not be predictable. With this conclusion a long lived misconception died and a new area of research which now bears the name Chaos was established. We intend to use some of the ideas from this new field to try to develop a tool for computational kinematics problems.

1.1 Computational kinematics

Computational kinematics is a vast field. Many problems require long and tedious computations. In the recent history three main areas evolved as major tools for dealing with these problems: matrix methods in iterative kinematic analysis, continuation methods, and algebraic elimination in mechanisms and robotics (Gupta[1993]). Of these three continuation methods hold a prominent position because they are developed to solve for all solutions of kinematics problems. They represent the state of the art of the subject. First application was developed by Roth and Freudenstein [1963a, 1963b]. Some difficulties, such as divergence or bifurcation were treated by Allgower and Georg[1980], Garcia and Zangwill [1981] and Morgan [1987]. The first application of polynomial continuation in mechanism problems was the work of Tsai an Morgan [1985]. Subsequent improvements follow in Morgan and Sommese [1987a, 1987b] and Wampler and Morgan [1989].

In this paper we depart from the ideas of these researches and take a new direction in finding all solutions to kinematic problems. For now our scope is limited to one variable nonlinear equation.

1.2 Stepping into computational chaos

A significant portion of the field of computational kinematics and geometry may be categorized as finding the roots of systems of highly nonlinear and transcendental equations. These equations represent time-varying geometric constraints of mechanical systems and their closed form solutions are often tedious and rare. Therefore, the most common method of solving is to employ numerical procedures based on the localized differential characteristics of the system (Gupta [1993]). However, these numerical procedures are generally inefficient in finding all the roots of the nonlinear equations. Their efficiency depends on an initial guess that has to be supplied and they yield only some of the solutions to the system. This problem has not been completely resolved to this date and it continues to attract researchers' attention (Raghavan and Roth [1995]). Resolution of this problem would also bring resolution to the problem of global optimization. Because, to try to solve for global optimum is to try to solve for all the roots of the system of equations and evaluate the cost function at these roots. We propose here to deal with these issues by using a new mathematical theory - chaos theory - and the Newton-Raphson numerical technique.

One of the first (and most popular) methods devised to numerically solve for the roots was the Newton-Raphson technique. This simple second order method shows superior performance when used to solve for the roots of the system of nonlinear equations. However, the literature offers innumerable examples of the cases where this numerical procedure runs into difficulties. Researchers not being able to explain these phenomena have referred to this difficulty as "algorithmic singularity" (Baillieul [988]), "artificial singularity" (Baker and Wampler [1988]) or "unavoidable singularity." Problems encountered are due to sensitive dependence on initial conditions of the Newton-Raphson method. But these difficulties can be useful and we intend to show that they provide information about all the roots of a nonlinear equation.

One drawback of the Newton-Raphson method can be stated as a question, "Which points should be used as initial guesses to obtain all the roots of the equations?" Our aim is to show that because of its iterative nature the method inherently contains the answer to that question. The areas of good initial guesses are those where the Newton-Raphson method shows numerical instability. These areas contain arbitrarily small copies of the **whole** domain which ensures converging to **all** the roots. This is the crucial observation that has its roots in the study of fractals (Mandelbrot [1982]) and we intend to present here how it can be used to resolve problems discussed above.

We first introduce the idea through an appropriate example. Then, we apply the Newton-Raphson method and provide a critical observation. Third, we present a method which searches for the regions of structural instabilities. Finely, we test our method on two examples.

1.3 Defining Chaos and Fractals

Before proceeding to the results, it is necessary to define some of the terms involved in this research. Definitions are taken from Srtogatz [1994].

Chaos *is a-periodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.*

Fractals *are complex geometric shapes with self-similar structure at arbitrarily small scales.*

2. CHAOTIC INSTABILITIES

With the birth of computers computational procedures invented a long time ago obtained a new look. One of these procedures called the Newton-Raphson method has became one of the main tools for solving nonlinear equations. It is one of many that is based on iteration. Through this process of iteration the solutions of nonlinear equations are determined. Even though this method is simple recent findings of the theory of Chaotic Dynamical Systems showed that the process of iterating of nonlinear functions produces very complex mathematical objects called fractals. Among many puzzling properties that these object posses the most useful one is that they are **self-similar** on arbitrary small scales (Madelbrot [1981]). In other words, by zooming into some regions of the domain of iteration we continue to see similar geometrical shapes that do not depend on the depth of our zooming. This property is the crucial one that enables finding all the roots of nonlinear equations. It is intrinsically connected to sensitivity of iterating process which we explain in the following section.

2.1 A closer look at the Newton-Raphson method

In Jovanovic and Kazerounian [1995] we described sensitivity of the Newton-Raphson method. Here we briefly repeat the main points and describe the key observation.

The Newton-Raphson method is a simple feedback scheme $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ which can be depicted in Figure 1.

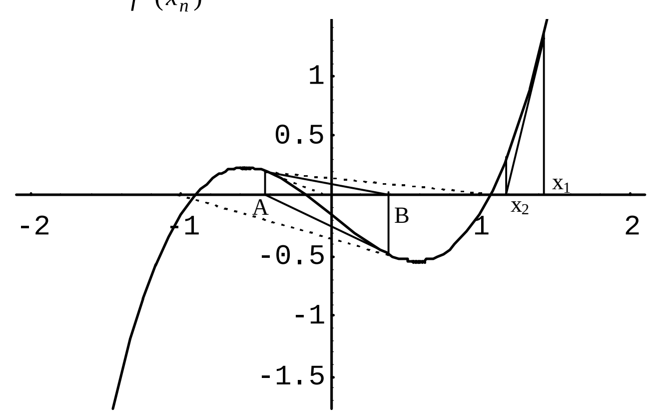

Figure 1. Graph of 3rd order polynomial with real roots.

By locally approximating $f(x)$ at x_1 with the tangent line and finding its intersection x_2 with the axis x for the next local approximation, Newton-Raphson rapidly converges to a "nearest" root. Intuition suggests that there are two points, A and B, for which the iterative formula would not converge. Rather, it would stay trapped between these two points because

$A = B - \frac{f(B)}{f'(B)}$ and $B = A - \frac{f(A)}{f'(A)}$. In chaos theory this is labeled as a period two orbit, because the outcome of the feedback scheme is the set of two repeatable points {*A, B, A, B,* ...}. Now, if the period two orbit is unstable for the iterative scheme, starting from vicinity of *A* or *B* the iterative process must "decide" to which root it will converge. And, as Figure 1 shows (by drawing slopes at *A* and *B*), all the roots are "visible" from points *A* and *B*. This leads to a conclusion that by locating unstable period points and using their vicinity as an initial point for the iteration, a "gate" to all the other real roots will be obtained. This conclusion we intend to utilize in section **3** to lay out a method for finding all the roots.

2.2 A simple example

To better demonstrate the above conclusion we use the example from Becker and Dorfler [1989]. It is required to find zeros of equation (1).

$$f(x) = x^3 - x = 0 \tag{1}$$

For this simple third order polynomial the zeros are $x_1 = -1, x_2 = 0, x_3 = 1$.

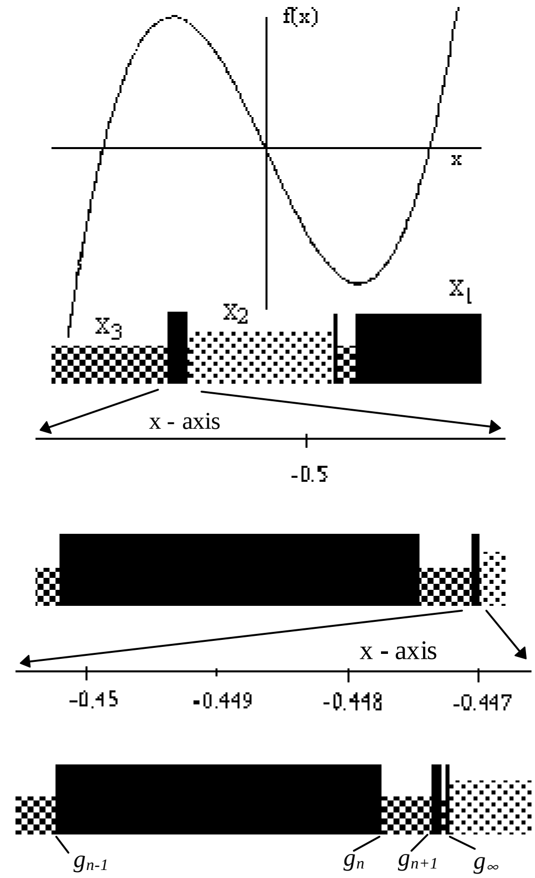

Figure 2. Basins of attraction.

Starting from any point on the real line in the Newton-Raphson iterative scheme converges to one of the zeros. These zeros have the nature of attractors, because every possible sequence of iterations tends to one of them. This behavior raises the question, "Given the starting value, which zero do we converge to?" More generally, defining a **Basin of Attraction** as a set of all initial points that lead to either of solutions, what are the basins of attraction for x_1, x_2, x_3? Figure 2 provides the answer. It shows the function $f(x)$ and three areas with a different pattern and height which represent basins of attraction for zeros x_1 , x_2 and x_3. They are plotted by determining the fate of each point on the real line under Newton-Raphson iteration. For example, we assign the mid-height pattern to a point on the real line for which iteration converges to zero x_2 .

Succeeding magnifications show self-similarity in the shape of basins of attraction on smaller scales. Patterns with this property are labeled as fractals. Furthermore, distribution of points g_n on the x - axis which separate the basins from each other reveals

$$\lim_{n\to\infty} \frac{g_n - g_{n-1}}{g_{n+1} - g_n} = 6 . \tag{2}$$

Expression (2) can be proved using the Mean Value Theorem and it suggests that there is an order in above rapidly interchanging zones. We also have that

$$\lim_{n\to\infty} g_n = -\frac{1}{\sqrt{5}}. \tag{3}$$

This limit provides us with a set of good starting values for the Newton-Raphson method. It actually is a point that belongs to a period two orbit as we discussed in section **2.1**. Therefore, interval $[-\frac{1}{\sqrt{5}}-\varepsilon, -\frac{1}{\sqrt{5}}+\varepsilon]$, where ε is an arbitrarily small positive number, contains copies of basins of attraction for $f(x)$ that exist on much larger scale . For that reason we know that by choosing initial values from this interval we will locate all the roots of $f(x)=0$. Now, it only remains to divide the interval into finitely many points and determine the fate of each point under the Newton-Raphson iteration. This will locate a certain number of roots or possibly all of them depending on the number of divisions which we will not discuss in this paper.

The above discussion is just an illustration of a theorem which states that a point on the boundary of one of the domains of attraction must be on the boundary of all of them (Falconer [1990]). This theorem enables us to find all the roots of an equation and consequently all design solutions.

2.3 More degrees of freedom

Consider a two-link manipulator in Figure 3. The behavior of this mechanism is completely determined by two simple trigonometric formulas.

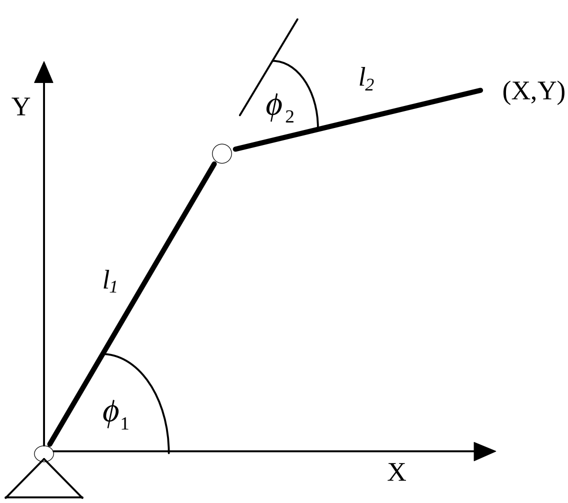

Figure 3. 2-Link manipulator.

Its kinematic equations can be transformed into an iterative formula by the Newton-Raphson technique. This formula is

$$\hat{\phi}_{n+1} = \hat{\phi}_n + J^{-1}(\hat{x}_d - \hat{x}_k) \tag{4}$$

where J is a nonlinear transformation that relates angular velocities of the linkages with the velocities of the tip of the manipulator, and $\hat{x}_d$ is a desired position of the manipulator. For given x and y there are two possible final states (I and II) of the iterative scheme as it is shown in the Figure 4.

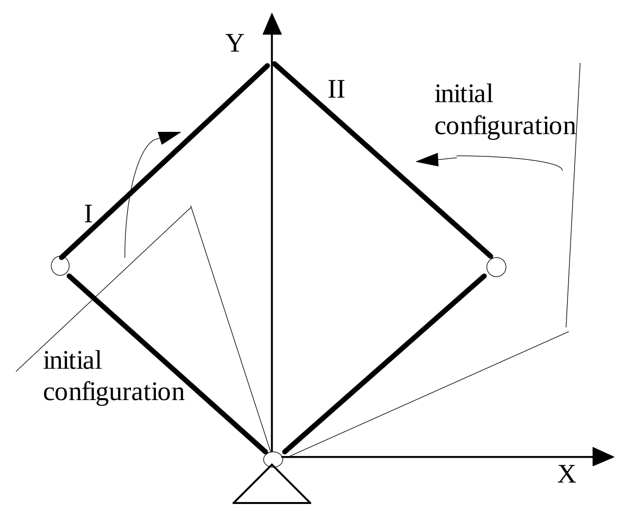

Figure 4. Two desired configurations.

We assign a color to each point depending on the final destiny of that point under iteration (4). Figure 5 represents the work space of the manipulator for $l_1=l_2=10$. It reveals an interesting fractal drawing with three distinct areas. Two of them (shown in different color) are the basins of attraction of two final (desired) positions of the manipulator. Blank areas represent the initial points for which the iteration did not converge. Self-similarity reveals itself again. Zooming will show similar patterns that are intrinsically dependent on the computational methods as well as the physical properties of the manipulator such as: singularities, Jacobian, etc.

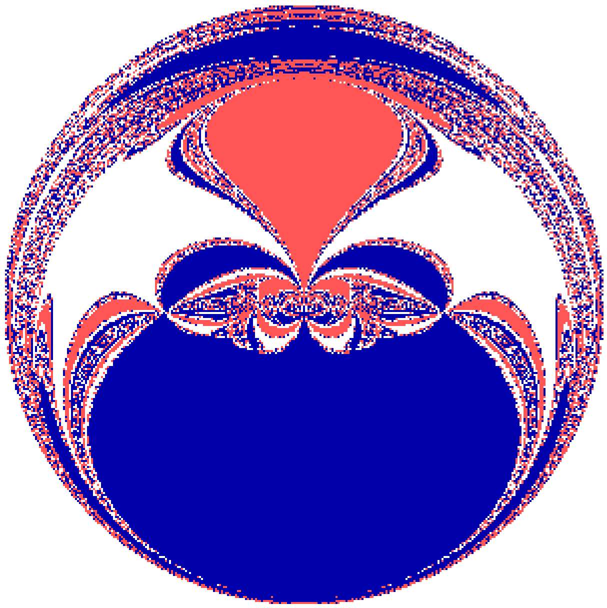

Figure 5. The basins of attraction for 2-link manipulator.

Even though the boundary between basins of attraction exhibits self-similarity it is not a fractal curve strictly speaking. The reasons for this are explained in Jovanovic [1997].

3. SEARCH FOR THE GATES TO ALL THE ROOTS

In the previous section we discussed areas of convergence for the Newton-Raphson method. We showed that boundaries of basins of attraction are fractal-like regions. These boundaries are sets of points that posses one very useful property which states that if a point is on the boundary of one basin of attraction is on the boundary of all of the them (Falconer [1991]). This property was established by Julia [1918] and Fatou [1919] and sets that posses it are called Julia sets. Therefore, we need to look for fractal-like sets that posses this property, but first we need a few fundamental definitions from the theory of Chaotic Dynamical Systems.

We denote the composition of two functions by $f\,o\,g(x) = f(g(x))$. If n-fold composition of f with itself recurs over and over again in the sequel, we denote this function by $f^n(x) = f\,o \dots o\,f(x)$. Note that superscript n does not represent derivative or power. Further, a point x is a fixed point for f if $f(x) = x$, and point x is a periodic point of period n if $f^n(x) = x$. The forward orbit of x is the set of points $x, f(x), f^2(x), \dots$ and it is denoted by $O^+(x)$. Let x_0 be a periodic point of period n, then $\lambda_{x_0} = (f^n)'(x_0)$ is the eigenvalue of the periodic orbit.

A periodic orbit $O^+(x_o)$ is

1. attracting if $0 < \left|\lambda_{x_0}\right| < 1$ and x_0 is called an attracting periodic point
2. superattracting $\lambda_{x_0} = 0$ and x_0 is called a superattracting periodic point
3. repelling if $\left|\lambda_{x_0}\right| > 1$ and x_0 is called a repelling periodic point
4. neutral if $\left|\lambda_{x_0}\right| = 1$ and x_0 is called a neutral periodic point

Of interest to us are the first three definitions. If x_0 is attracting or superattracting periodic point then there is an open interval U about x_0 such that if $x \in U$, then $\lim_{n\to\infty} f^n(x) = x_0$. If x_0 is repelling periodic point then $\lim_{n\to\infty} f^n(x) \neq x_0$ unless $x = x_0$. Note that a fixed point is a periodic point with period one. For complete discussion refer to Devaney[1989].

Now, in view of all this Newton-Raphson method represents iteration of a function $N(x) = x - \frac{f(x)}{f'(x)}$ or $x_{k+1} = N(x_k)$. Further, roots of $f(x)$ are superattracting fixed points of N. This can be seen by differentiating $N(x)$ at a root.

$$N'(x)\Big|_{x=p} = 1 - \frac{f'(x)^2 - f(x)\,f'(x)}{f'(x)^2}\Bigg|_{x=p} = 1 - 1 + f(p)\,f'(p) = 0 \tag{5}$$

since $f(p)=0$ where p is any root of f. This is why Newton-Raphson method is quadratically convergent method.

3.1 Iterating inverse function

In section **2.1** we showed that if a real function has more then one real root there exists some region which is in the vicinity of unstable (repelling) periodic points from which we can reach all the roots of the function. In other words, by locating unstable periodic points of the function in question we locate the region which contains "good" set of initial points to be used to find all the roots of $f(x)$. Further, being a fractal this set in a small vicinity of unstable periodic points contains the "picture" of all the roots of $f(x)$ on much larger scale.

Unstable periodic points play a crucial role here because all of them belong to Julia sets which are known to be fractals. Therefore our primary goal is to find one of unstable periodic points of N. The simplest possible is the periodic point with period 2. Mathematically, it means solving $N^2(x) = x$ for x with the condition $\left|(N^2)'(x)\right| > 1$. For this nonlinear equation a number of numerical techniques are available. One of them is, again, Newton-Raphson which in this case is inadequate because most of initial points would lead to stable points of $N(x)$ (roots of $f(x)$), since these points are also fixed points of $N^2(x)$. Other methods are also not appropriate because initial numerical choice always needs to be made with requiring above condition. For this reason we look for the procedure which would take us to the sought region from one of the roots of $f(x)$ since we can easily locate at least one root by applying Newton-Raphson. In other words we locate one root from which we move to the boundary of the basins of attraction.

Consider $x_{k+2}=N^2(x_k)$.

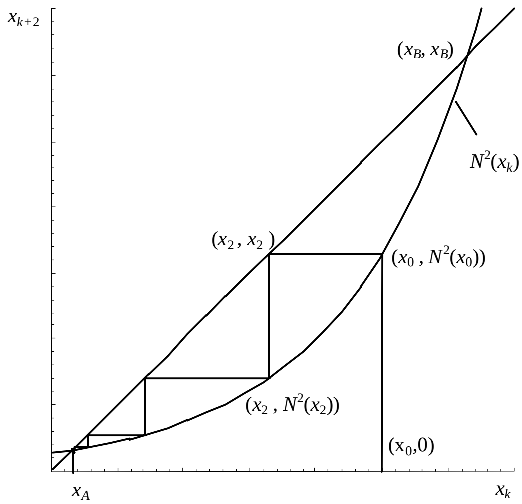

Figure 6. Graphical representation of iterating process.

Iterating graphically $x_{k+2}=N^2(x_k)$ leads to the construction of a cobweb depicted in Figure 6. The cobwed is a set of connected line segments beginning at x_0. It consists of one vertical line segment from $(x_0,0)$ to $(x_0, N^2(x_0))$ and one horizontal line segment from $(x_0, N^2(x_0))$to (x_2, x_2). Its further parts also consist first of one vertical and then one horizontal line segments which all end after infinitely many iterations in the fixed point.

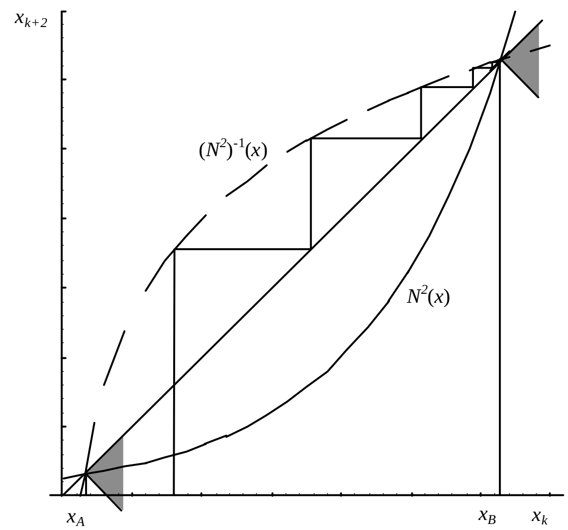

Figure 7. Graphically iterating inverse function.

This reveals that point x_A is attractive and x_B is repelling under N^2 , because $\left|(N^2)'(x_A)\right| < 1$ (vicinity of $N^2(x)$ around x_A is within shaded region, Figure 7) and $\left|(N^2)'(x_B)\right| > 1$ (vicinity of $N^2(x)$ around x_B is outside of shaded region). Our aim is to find a way to reverse attractiveness of these two points.

The dashed curve in Figure 7 represents the mirror image of $N^2(x_k)$ about $x_{k+2}=x_k$ and it is actually the inverse function of N^2 which we denote $(N^2)^{-1}$. This function exists on some segment $[a,b]$ if N^2 is decreasing or increasing in that segment. We assume now that such function exists in segment $[x_A-h, x_B-h]$ where $h>0$. Graph of $(N^2)^{-1}$ shows that whatever parts of N^2 were in the shaded region are now out of it and vice versa. This means that under $(N^2)^{-1}$ x_A and x_B should reverse their stability. To check this we differentiate $(N^2)^{-1}$ at x_A and x_B with knowing that $N^2(x_A)=x_A$ and $N^2(x_B)=x_B$ are satisfied. For brevity in notation we denote $g\equiv(N^2)$ and $h\equiv(N^2)^{-1}$.

Beginning with

$$h(g(x)) = x \tag{6}$$

and differentiating both sides

$$\frac{d}{dx}h(g(x)) = g'(x)\frac{d}{dg}h(g(x)) = 1 \tag{7}$$

we obtain

$$\frac{d}{dg}h(g(x)) = \frac{1}{g'(x)} \tag{8}$$

At x_A with $g(x_A)=x_A$ we have

$$\frac{d}{dg}h(g(x_A)) = \frac{d}{dx_A}h(x_A) = \frac{1}{g'(x_A)} \tag{9}$$

Now for x_A to be an attractive point under $h(x)$ $\left|\frac{1}{g'(x_A)}\right| < 1$ should be satisfied which is the case since $|g'(x_A)| = |(N^2)'(x_A)| > 1$ from before. The same reasoning holds for x_B.

Therefore, with simple iteration of inverse function the stability of the existing fixed points is reversed. Such iterating secures approaching the boundary of the basins of attraction starting from any root of the equation in question.

To summarize, we first iterate by Newton-Raphson method until we get one "closest" root, and then use this root to iterate backwards to get to the boundary of basins of attraction. Once we get there we use points in the vicinity of this boundary as a "good" initial set of points for Newton-Raphson method. Since this set of points is fractal in nature under Newton-Raphson iteration we are guaranteed to find all roots of the equation.

3.2 Using inverse function for iteration

For existence of inverse function it is necessary that the original function be increasing or decreasing. This condition is certainly not satisfied for $N(x)$ and its n-folded composition. Therefore, it is only reasonable to talk about local invertibility of the functions. In other words, given $f{:}R\rightarrow R$ such that $f(a)=b$ and $f'(a)\neq 0$ there exists neighborhood $U=[a-\delta,a+\delta]$ of a and $V=[b-\delta,b+\delta]$ of b such that given $y_* \in V$, the sequence $\{x_n\}_0^\infty$ defined inductively by

$$x_0 = a, \;\; x_{n+1} = x_n + \frac{y_* - f(x_n)}{f'(a)} \tag{10}$$

converges to a (unique) point $x_* \in U$ such that $f(x_*)=y_*$. In short, equation (10) establishes existence of the inverse function in some region V. Therefore inverse function of f has a unique representation in V and can be expanded in Taylor series in x at b where $x\in V$. So we have

$$\varphi(x) = \varphi(b) + \varphi'(b)(x-b) + \frac{\varphi''(b)}{2}(x-b)^2 + \ldots \tag{11}$$

and if $b=f(x)$ we obtain

$$\varphi(x) = x + \varphi'(f(x))(x - f(x)) + \frac{\varphi''(f(x))}{2}(x - f(x))^2 + \ldots \tag{12}$$

Now, referring to previous section we have an iterative formula $x_{k+1}=\varphi(x_k)$ to use in search for unstable periodic orbits. Note that derivatives in equation (12) are not on x. They are derivatives on the argument of φ which is codomen of f. We obtain them by successive differentiation of

$$\varphi(f(x)) = x \tag{13}$$

yielding

$$\begin{aligned} &\frac{d}{df}\varphi(f(x)) = \frac{1}{f'(x)}, \\ &\frac{d^2}{df^2}\varphi(f(x)) = \frac{1}{f'(x)}\frac{d}{dx}\left(\frac{1}{f'(x)}\right), \\ &\frac{d^3}{df^3}\varphi(f(x)) = \frac{1}{f'(x)}\frac{d}{dx}\left(\frac{1}{f'(x)}\frac{d}{dx}\left(\frac{1}{f'(x)}\right)\right), \\ &\ldots\ldots \end{aligned} \tag{14}$$

Although equation (12) is an exact inverse function of f, in this form it is not practical for inversion of N^2. This is because all the terms in the derivatives have $f'(x)$ in the denominator which causes $\varphi(x)$ to have a very large value close to the stable fixed points of N^2. These points are superattracting and by definition the slope of N^2 at the points is zero. Division by zero causes the formula to blow up. Therefore, it seams that Newton-Raphson's characteristics of fast convergence creates a constraint for our method.

One way to deal with this problem is to remove superattractiveness of fixed points of N^2 without moving fixed points themselves. In other words we need only change the slope of N^2 at all fixed points. This can be easily done by using a control parameter $0<\mu<1$. In section **3.1** we showed that $\left.\frac{d}{dx}N(x)\right|_{x=p} = 0$. It can also be shown that $\left.\frac{d}{dx}N^n(x)\right|_{x=p} = 0$ for all n positive integers. Consider now $x_{k+2}=N^2(x_k)$. Adding x_k-x_k to the right hand side we obtain $x_{k+2}=x_k-x_k+N^2(x_k)$. Now last two terms are a step change in iteration from value x_k to x_{k+2}. Therefore, by cutting the step with μ so that $x_{k+2}=N^{*2}(x_k)\equiv x_k-\mu(x_k-N^2(x_k))$ we obtain a new function that has the same fixed points and controllable slope at these points. To check this we first substitute a fixed point p into the expression to verify that $p=p-\mu(p-N^2(p))$. This is true since $p=N^2(p)$ so the fixed points of N^{*2} are the same as those of N^2. Second, differentiating $N^{*2}(x_k)$ we get

$$\left.\frac{d}{dx}N^{*2}(x)\right|_{x=p} = 1-\mu(1-\left.\frac{d}{dx}N^{2}(x))\right|_{x=p} = 1-\mu \quad (15)$$

Therefore we can control the slope of N^2 at fixed points via parameter μ. Consequently, inverting N^{*2} to find $\varphi^*(x)$ we obtain an iterative formula $x_{k+2}=\varphi^*(x_k)$ where in $\varphi^*(x_k)$ it is sufficient to use first two terms in Taylor series expansion if μ is small enough. Finally we obtain

$$x_{k+2} = x_k + \frac{x_k - N^{*2}(x_k)}{(N^{*2})'(x_k)} \quad (16)$$

Choice of parameter μ involves sensitivity analysis which we will not discuss in this paper. Now, all the tools for obtaining a good set of initial points for Newton-Raphson method are available.

In summary, we first construct $N^2(x_k)$ and locate one of the roots of $f(x)$. Then we use $N^{*2}(x_k)$ to construct $\varphi^*(x_k)$ and iterate it with the root as initial seed. This iteration should converge to an unstable periodic point in which vicinity is our good set of initial points to solve for the roots of the original equation. Note that it might happen that the inverse search diverges to infinity. This occurs sometimes because infinity is a repelling point (with any period) of the polynomial and inverse iteration is correctly finding it. This should not happen often unless there are very few real roots of the polynomial and μ is too big or the root from which we start iterating is the biggest or smallest of all other roots. In such cases some other roots should be selected for inverse iterating.

4. NUMERICAL EXPERIMENT

In section **3** we developed a method that locates the regions of structural instability of the Newton-Raphson technique. In this section will test the method on two examples and locate the "gates to all the roots". Then we will plot the basins of attraction map in the vicinity of these gates. We should be able to locate all the roots if our interval division is fine enough.

Consider polynomial (1) from section **2** again. Its roots are $x_1 = -1, x_2 = 0, x_3 = 1$. Using equation (16) yields an iterative formula that is too complex to be displayed here. However, its numerical implementation is more desirable since it is very easily programmed. Starting inverse iteration (16) with root $0 \pm \varepsilon$ where ε is an arbitrarily small positive number leads to period 2 unstable points ± 0.4472136 (which is $\pm\frac{1}{\sqrt{5}}$). Compare these points with the limit in equation (3). Therefore, we located two points that lie on the boundary to all basins of attraction. Plotting basins of attraction map in the vicinity of these points would generate Figure 2. Note that inverse iterating from the other two roots ± 1 leads to infinity because these are the biggest and the smallest roots of the polynomial.

Consider now an eight order polynomial

$$f(x) = (x-1)(x-3)(x^2-4)(x^2+9)(x^2+16) \quad (17)$$

Roots of this polynomial are x_1=1, $x_{2/3}$=±2, x_4=3, $x_{5/6}$=±3i, $x_{7/8}$=±4i. Since not all the roots are real the Newton-Raphson method is not guarantied to converge everywhere on the real axis. However, areas of non convergence represent just another type of a basin of attraction and they too have to obey the fractal pattern. Now, we proceed again with iterative search for the gates. Starting from roots 1 and 2 we find all the unstable period 2 points. Results are displayed in Table 1 .

1-ε	1+ε	2-ε	2+ε
-.993959	1.457301	1.62807	2.54079

Table 1. Unstable period 2 points.

The other real roots lead to infinity and consequently they are discarded. Here for demonstration purpose we found all of the unstable period 2 points. However, this was not necessary. Finding anyone of them would be the end of our search because the vicinity of anyone of them contains the picture of the basins of attraction for the whole domain. This is shown in Figure 8. Each strip represents the vicinity of the gates and each color represents a basin of a root. Observing four different gray scales we know that we detected all the real roots of the polynomial which means that our interval division was fine enough.

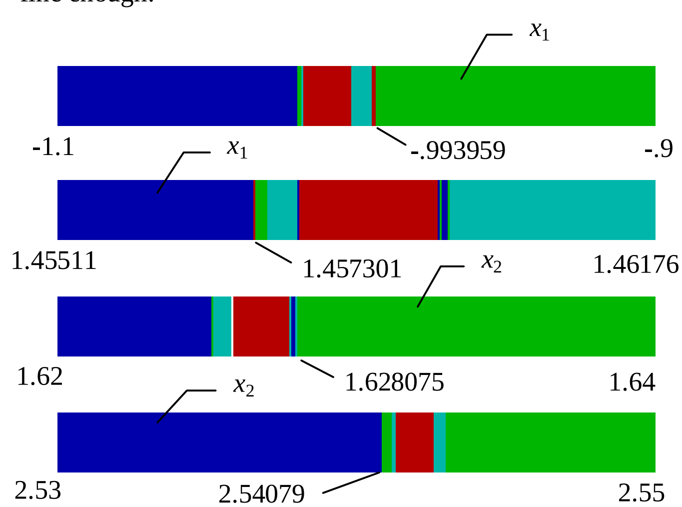

Figure 8. Basins of attraction in the vicinity of unstable period 2 points.

Note in Figure 8 that there are no nonconverging areas except in the 3rd strip (a tiny white portion). Note that each strip has its own assignment of colors for the roots.

4. CONCLUSIONS

We showed that the Newton-Raphson method contains information about all the roots of the equation in question. Such information is stored in the arbitrarily close vicinity of geometrical objects that separate basins of attraction. These objects called fractals are the “gates” to all the roots of a nonlinear equation, because they provide a good set of initial points for the Newton-Raphson method. We presented a method that locates these gates and consequently enables finding all the roots of the equation.

The work presented here is just a small step forward in using the recently developed concept of chaos. We have tried to use one of the findings in this exciting area, the property from section **3**. The method we developed is just a beginning and it is not the only way to utilize this property. There are other ways such as introducing structural instability over the whole domain so that we do not have to “look” for it or artificially adding known roots to the equation and consequently controlling the shape of basin of attraction. These ideas are still in their infancy and we will report on their development. However, our immediate concern is to extend our sensitivity inverse method to a set of polynomial equations with more variables. Such progress would enable us to deal directly with issues of computational kinematics.

5. REFERENCES

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